MEC-E8003 Beam, Plate and Shell models, exam 27.05.2019

1. Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations of the cylindrical coordinate system

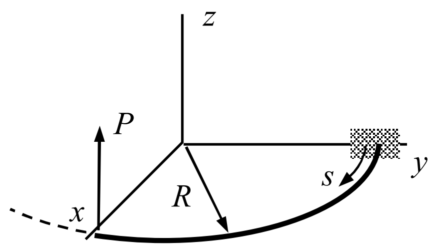
$$\vec{r}=\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix},\quad \nabla=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \partial/\partial r \\ \partial/(r\partial\phi) \\ \partial/\partial z \end{Bmatrix} \text{ and } \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix} \text{ (zeros otherwise).}$$

2. Virtual work expression of a linearly elastic bar supported by a spring at the right end $x=L$ $(n=1)$ is given by

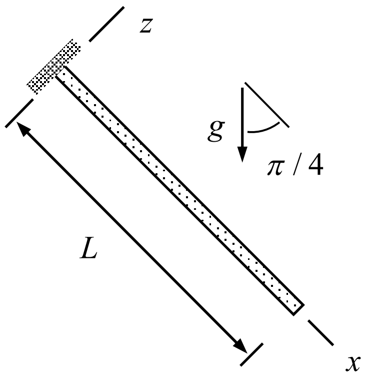
$$\delta W=\int_0^L -(EA\frac{du}{dx}\frac{d\delta u}{dx})dx+\int_0^L (b\delta u)dx-(ku\delta u)_{x=L},$$

in which $EA=EA(x)$ and k, b are constants. Displacement vanishes at the left end $x=0$ $(n=-1)$ of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W=0$ $\forall\delta u\in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x=0$.

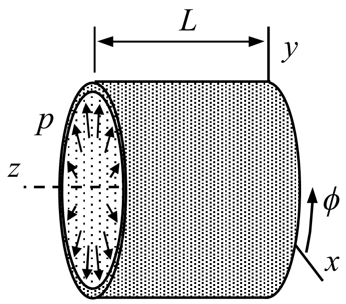
3. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the (s,n,b)–coordinate system.



4. Consider a cantilever Reissner-Mindlin plate strip (long in the y–direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx}=N(x)$, $Q_x=Q(x)$, $M_{xx}=M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



5. A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry. Young's modulus E and Poisson's ratio ν of the material are constants.



Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations of the cylindrical coordinate system

$$\vec{r}=\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix},\quad \nabla=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \partial/\partial r \\ \partial/(r\partial\phi) \\ \partial/\partial z \end{Bmatrix} \text{ and } \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix} \text{ (zeros otherwise).}$$

Solution

In a term, gradient operator ∇ acts on everything on its right hand side. Otherwise the operator is treated like a vector (if the basis vectors are not constants, the derivative operators should be after the basis vectors)

$$\vec{r}=\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}=r\vec{e}_r+z\vec{e}_z,$$

$$\nabla=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \partial/\partial r \\ \partial/(r\partial\phi) \\ \partial/\partial z \end{Bmatrix}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}.$$

Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term by term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.
2p Gradient of the position vector is a second order tensor

(I) $\nabla\vec{r}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z})(r\vec{e}_r+z\vec{e}_z)\quad\Leftrightarrow$

(II) $\nabla\vec{r}=\vec{e}_r\frac{\partial}{\partial r}r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}z\vec{e}_z+\vec{e}_\phi\frac{\partial}{r\partial\phi}r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}z\vec{e}_z+\vec{e}_z\frac{\partial}{\partial z}r\vec{e}_r+\vec{e}_z\frac{\partial}{\partial z}z\vec{e}_z\quad\Leftrightarrow$

(III) $\nabla\vec{r}=\vec{e}_r\vec{e}_r+0+\vec{e}_\phi\vec{e}_\phi+0+0+\vec{e}_z\vec{e}_z\quad\Leftrightarrow$

(IV) $\nabla\vec{r}=\vec{e}_r\vec{e}_r+\vec{e}_\phi\vec{e}_\phi+\vec{e}_z\vec{e}_z=\vec{I}$. ←

2p Divergence of the position vector is a scalar

(I) $\nabla\cdot\vec{r}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z})\cdot(r\vec{e}_r+z\vec{e}_z)\quad\Leftrightarrow$

(II) $\nabla\cdot\vec{r}=\vec{e}_r\frac{\partial}{\partial r}\cdot r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}\cdot z\vec{e}_z+\vec{e}_\phi\frac{\partial}{r\partial\phi}\cdot r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}\cdot z\vec{e}_z+\vec{e}_z\frac{\partial}{\partial z}\cdot r\vec{e}_r+\vec{e}_z\frac{\partial}{\partial z}\cdot z\vec{e}_z\quad\Leftrightarrow$

(III) $\nabla\cdot\vec{r}=1+0+1+0+0+1\quad\Leftrightarrow$

(IV) $\nabla\cdot\vec{r}=3$. ←

2p Curl of the position vector is a vector

(I) $\nabla\times\vec{r}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z})\times(r\vec{e}_r+z\vec{e}_z) \quad \Leftrightarrow$

(II) $\nabla\times\vec{r}=\vec{e}_r\frac{\partial}{\partial r}\times r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}\times z\vec{e}_z+\vec{e}_\phi\frac{\partial}{r\partial\phi}\times r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}\times z\vec{e}_z+\vec{e}_z\frac{\partial}{\partial z}\times r\vec{e}_r+\vec{e}_z\frac{\partial}{\partial z}\times z\vec{e}_z \quad \Leftrightarrow$

(III) $\nabla\times\vec{r}=0+0+0+0+0+0 \quad \Leftrightarrow$

(IV) $\nabla\times\vec{r}=0$. ←

Virtual work expression of a linearly elastic bar supported by a spring at the right end $x=L$ $(n=1)$ is given by

$$\delta W=-\int_0^L (\frac{d\delta u}{dx}EA\frac{du}{dx})dx+\int_0^L (\delta u b)dx-(\delta u k u)_{x=L},$$

in which $EA=EA(x)$ and k, b are constants. Displacement vanishes at the left end $x=0$ $(n=-1)$ of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W=0\ \forall \delta u\in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x=0$.

Solution

Fundamental theorem of calculus (integration by parts) and the fundamental lemma of variation calculus are the tools for deriving a boundary value problem starting from a virtual work expression. In the one-dimensional case, for any continuous functions a and b (or values at some point), it holds

$$\int_\Omega a\frac{db}{dx}dx=\sum_{\partial\Omega}(nab)-\int_\Omega \frac{da}{dx}bdx \quad (\text{where } n=\pm 1),$$

$$a,b\in\mathbb{R}: \qquad ab=0 \quad \forall b \quad \Leftrightarrow \quad a=0,$$

$$a,b\in C^0(\Omega): \quad \int_\Omega abdx=0 \quad \forall b \quad \Leftrightarrow \quad a=0 \quad \text{in} \quad \Omega.$$

2p In the present case $\Omega=(0,L)$ and $\partial\Omega=\{0,L\}$. Displacement has continuous derivatives up to and including second order i.e. $u\in C^2(\Omega)$. The constraint on the function set $u=0$ at $x=0$ implies that $\delta u=0$ at $x=0$. Integration by parts gives equivalent forms (the aim is to remove the derivatives from variations in the integral over the domain)

$$\delta W=-\int_0^L (\frac{d\delta u}{dx}EA\frac{du}{dx})dx+\int_0^L (\delta u b)dx-(\delta u k u)_{x=L} \quad \Leftrightarrow$$

$$\delta W=\int_0^L [\frac{d}{dx}(EA\frac{du}{dx})+b]\delta u dx-[(EA\frac{du}{dx}+ku)\delta u]_{x=L}. \quad (\text{as } \delta u=0 \text{ at } x=0)$$

2p The purpose of the manipulation above was to obtain a representation which allows the use of fundamental lemma of variation calculus. According to principle of virtual work, $\delta W=0\ \ \forall \delta u\in U$. Let us consider first a subset $U_0\subset U$ for which $\delta u=0$ at $x=L$ so that the boundary term vanishes. Then

$$\delta W=\int_0^L [\frac{d}{dx}(EA\frac{du}{dx})+b]\delta u dx=0 \quad \delta u\in U_0\subset U$$

and the fundamental lemma of variation calculus implies that

$$\frac{d}{dx}(EA\frac{du}{dx})+b=0 \ \text{ in } (0,L).$$

2p Knowing this and considering the full set U, the variational equation simplifies into

$$\delta W = -[(EA\frac{du}{dx} + ku)\delta u]_{x=L} = 0 .$$

Then, the fundamental lemma of variation calculus implies that

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L .$$

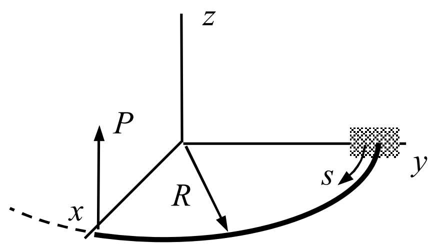
Finally combining the equations to form a boundary value problem (notice that the definition of the function set implies also a boundary condition):

$$\frac{d}{dx}(EA\frac{du}{dx}) + b = 0 \quad \text{in} \ (0, L) ,$$

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L ,$$

$$u = 0 \quad \text{at} \quad x = 0 .$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N'-Q_n\kappa+b_s \\ Q_n'+N\kappa-Q_b\tau+b_n \\ Q_b'+Q_n\tau+b_b \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n\kappa+c_s \\ M_n'+T\kappa-M_b\tau-Q_b+c_n \\ M_b'+M_n\tau+Q_n+c_b \end{Bmatrix}=0.$$

For a circular beam, curvature and torsion are $\kappa=1/R$ (constant) and $\tau=0$.

3p As external distributed forces and moments vanish i.e. $b_s=b_n=b_b=c_s=c_n=c_b=0$, equilibrium equations and the boundary conditions at the free end simplify to (notice that the external force acting at the free end is acting in the oppisite direction to $\vec{e}_b$)

$$\begin{Bmatrix} N'-Q_n/R \\ Q_n'+N/R \\ Q_b' \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n/R \\ M_n'+T/R-Q_b \\ M_b'+Q_n \end{Bmatrix}=0 \quad s\in\left]0,R\frac{\pi}{2}\right[,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b+P \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix}=0 \quad s=R\frac{\pi}{2}.$$

3p Equations constitute a boundary value problem which can be solved by hand calculations without too much effort;

$$Q_b'=0 \quad s\in\left]0,R\frac{\pi}{2}\right[\quad \text{and} \quad Q_b+P=0 \quad s=R\frac{\pi}{2} \quad \Rightarrow \quad Q_b(s)=-P. \quad \Leftarrow$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition give

$$N''+\frac{1}{R^2}N=0 \quad s\in\left]0,R\frac{\pi}{2}\right[\quad \text{and} \quad N'=N=0 \quad s=R\frac{\pi}{2} \quad \Rightarrow \quad N(s)=0 \quad \Leftarrow$$

The first equilibrium equation gives

$$Q_n(s)=0. \quad \Leftarrow$$

After that, continuing with the moment equilibrium equations with the solutions to the force equilibrium equations

$$M_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad M_b = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad M_b(s) = 0 \,.$$

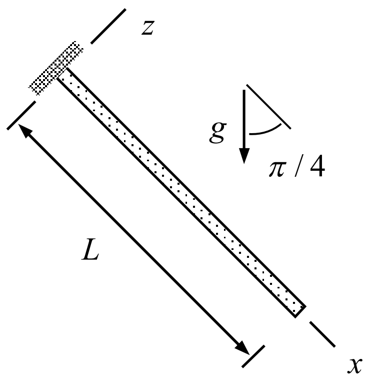
Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives

$$T'' + \frac{1}{R^2}T + \frac{P}{R} = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad T' = T = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad T = PR(\sin\frac{s}{R} - 1) \,.$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = RP\cos\frac{s}{R} \,.$$

Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx}=N(x)$, $Q_x=Q(x)$, $M_{xx}=M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



Solution

Equilibrium and constitutive equations of the thin-slab and bending modes are

$$\begin{Bmatrix} \frac{\partial}{\partial x}N_{xx}+\frac{\partial}{\partial y}N_{xy}+b_x \\ \frac{\partial}{\partial y}N_{yy}+\frac{\partial}{\partial x}N_{xy}+b_y \end{Bmatrix}=0,\quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \frac{\partial}{\partial x}Q_x+\frac{\partial}{\partial y}Q_y+b_n \\ \frac{\partial}{\partial x}M_{xx}+\frac{\partial}{\partial y}M_{xy}-Q_x \\ \frac{\partial}{\partial y}M_{yy}+\frac{\partial}{\partial x}M_{xy}-Q_y \end{Bmatrix}=0,\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{Bmatrix} \frac{\partial\theta}{\partial x}-\nu\frac{\partial\phi}{\partial y} \\ -\frac{\partial\phi}{\partial y}+\nu\frac{\partial\theta}{\partial x} \\ \frac{1-\nu}{2}(\frac{\partial\theta}{\partial y}-\frac{\partial\phi}{\partial x}) \end{Bmatrix} \text{ and } \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \frac{\partial w}{\partial x}+\theta \\ \frac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

4p Derivatives with respect to y vanish, $b_x=\rho g t/\sqrt{2}$, and $b_n=-\rho g t/\sqrt{2}$. The Reissner-Mindlin plate equations of the planar problem simplify to

$$\frac{dN}{dx}+\frac{\rho g t}{\sqrt{2}}=0,\quad \frac{dQ}{dx}-\frac{\rho g t}{\sqrt{2}}=0,\quad \frac{dM}{dx}-Q=0 \quad \text{in} \quad (0,L), \leftarrow$$

$$N=\frac{tE}{1-\nu^2}\frac{du}{dx},\quad Q=Gt(\frac{dw}{dx}+\theta),\quad M=D\frac{d\theta}{dx} \quad \text{in} \quad (0,L), \leftarrow$$

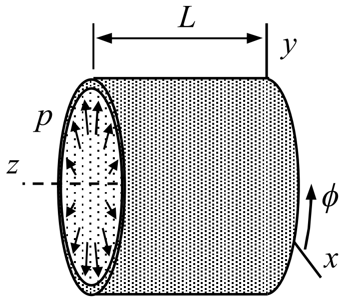
2p Boundary conditions can be deduced from the figure:

$$u=0,\quad w=0,\quad \theta=0 \quad \text{at} \quad x=0, \leftarrow$$

$$N=0,\quad M=0,\quad Q=0 \quad \text{at} \quad x=L. \leftarrow$$

Solution to equations can be obtained by considering the equilibrium equations and the boundary conditions at the free end first. After that, solutions to the displacement components follow from the constitutive equations and the boundary conditions at the clamped edge.

A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry and $u_\phi = 0$. Young's modulus E and Poisson's ratio ν of the material are constants.



Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \frac{1}{R}\frac{\partial}{\partial\phi}N_{z\phi}+\frac{\partial}{\partial z}N_{zz}+b_z \\ \frac{\partial}{\partial z}N_{z\phi}+\frac{1}{R}\frac{\partial}{\partial\phi}N_{\phi\phi}+b_\phi \\ \frac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{Bmatrix} \frac{\partial}{\partial z}u_z+\nu\frac{1}{R}(\frac{\partial}{\partial\phi}u_\phi-u_n) \\ \frac{1}{R}(\frac{\partial}{\partial\phi}u_\phi-u_n)+\nu\frac{\partial}{\partial z}u_z \\ \frac{1-\nu}{2}(\frac{1}{R}\frac{\partial}{\partial\phi}u_z+\frac{\partial}{\partial z}u_\phi) \end{Bmatrix}.$$

3p Due to the rotation symmetry, the derivatives with respect to the angular coordinate vanish and $u_\phi = 0$. External distributed force $b_n = -p$ is due to the traction acting on the inner boundary. Therefore, the equilibrium equations and constitutive equations simplify to a set of ordinary differential equations

$$\frac{dN_{zz}}{dz}=0, \quad \frac{dN_{z\phi}}{dz}=0, \quad \frac{1}{R}N_{\phi\phi}-p=0 \quad \text{in} \quad (0,L),$$

$$N_{zz}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz}-\nu u_n), \quad N_{\phi\phi}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz}-u_n), \quad N_{z\phi}=0 \quad \text{in} \quad (0,L),$$

As the edges are stress-free i.e.

$$N_{zz}=0 \text{ and } N_{z\phi}=0 \quad \text{on} \quad \{0,L\}.$$

3p Solution to the stress resultants, as obtained from the equilibrium equations, are

$$N_{zz}=0, \; N_{z\phi}=0, \text{ and } N_{\phi\phi}=Rp.$$

Constitutive equations give

$$N_{zz}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz}-\nu u_n)=0 \quad \Rightarrow \quad \frac{du_z}{dz}=\frac{\nu}{R}u_n \quad \text{and}$$

$$Rp=N_{\phi\phi}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz}-u_n)=\frac{tE}{1-\nu^2}\frac{1}{R}(\nu^2-1)u_n=-\frac{tE}{R}u_n \quad \Leftrightarrow \quad u_n=-\frac{pR^2}{tE}.$$