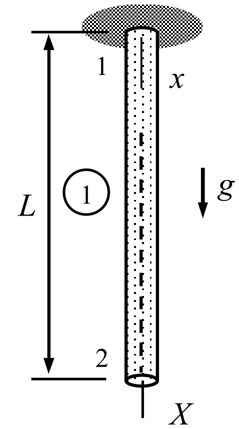
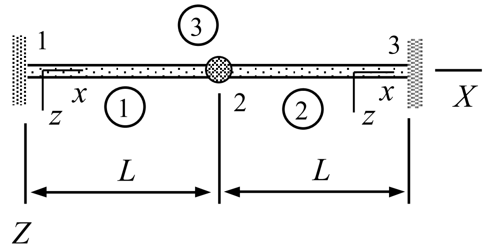
MEC-E8001 Finite Element Analysis, Exam 29.05.2019

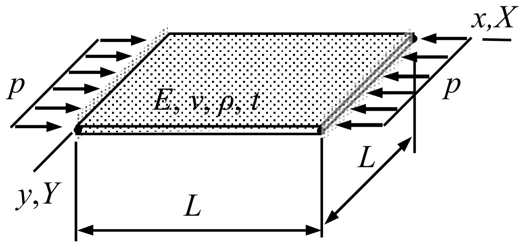
1. Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1-x/L)u_{x1} + (x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



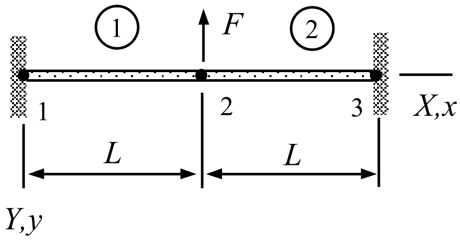
2. The XZ-plane structure shown consists of two *mass-less* beams and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of displacements u_{Z2} and θ_{Y2}. Young's modulus of the beam material and the second moment of area are E and I, and the mass and moment of inertia of the disk are m and J, respectively.



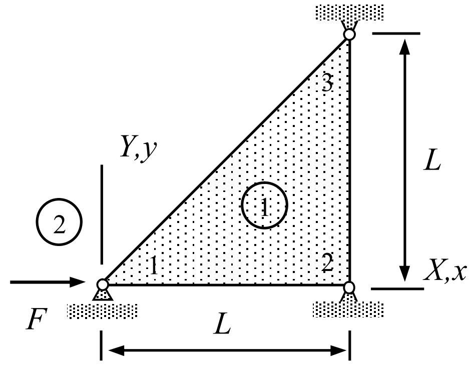
3. Determine the critical value of the in-plane loading p_{cr} making the plate of the figure to buckle. The loaded edges are simply supported and the unloaded free. Use the approximation $w(x,y) = a_0(1-x/L)(x/L)$ and assume that $N_{xx} = -p$ and $N_{yy} = N_{xy} = 0$. Problem parameters E, ν, ρ and t are constants.

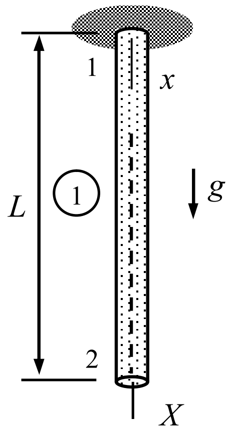


4. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2} = 0$). When $F = 0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = CE_{xx}$, in which C is constant. Use two elements with linear shape functions.



5. A thin triangular slab (assume plane stress conditions) loaded by a horizontal force is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. At the constant initial temperature ϑ° and loading $F = 0$, stress vanishes. If the slab is heated to the constant temperature $2\vartheta^\circ$, what is the required force F to have $u_{X1} = 0$? Material properties E, ν, α and thickness t of the slab are constants.





Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx)+\delta u f_x$ and approximation $u=(1-x/L)u_{x1}+(x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are just substituted into the density expression followed by integration over the domain occupied by the element (line segment, triangle etc.). Here the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x \quad \text{and} \quad u=(1-\frac{x}{L})u_{x1}+\frac{x}{L}u_{x2}.$$

2p The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \;\Rightarrow\; \delta u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix},$$

$$\frac{du}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \;\Rightarrow\; \frac{d\delta u}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

2p When the approximation is substituted there, virtual work density expression of the bar model takes the form

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x = -\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}+\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}-1\\ 1\end{Bmatrix}\frac{1}{L}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f) \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f)\,.$$

Finally, integration over the element gives the virtual work expression of the bar element

$$\delta W=\int_0^L \delta w_\Omega dx = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{fL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}).$$

2p Finding the displacement of the free end follows the usual lines. Here, $f_x = \rho g A$, $u_{x1} = u_{X1} = 0$, and $u_{x2} = u_{X2}$

$$\delta W = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{\rho g A L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right) = -\delta u_{X2} \left(\frac{EA}{L} u_{X2} - \frac{\rho g A L}{2} \right) = 0 \quad \forall \delta u_{X2} \quad \Leftrightarrow$$

$$\frac{EA}{L} u_{X2} - \frac{\rho g A L}{2} = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{\rho g L^2}{2E}.$$

One may start also from approximation which takes into account the boundary condition and solve the problem with steps

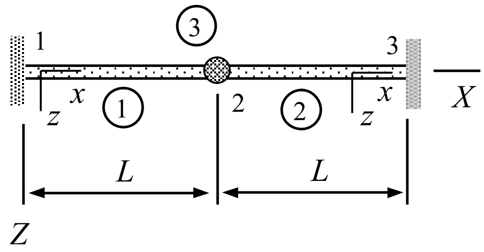
$$u = \frac{x}{L} u_{X2}, \quad \delta u = \frac{x}{L} \delta u_{X2}, \quad \frac{du}{dx} = \frac{1}{L} u_{X2}, \quad \frac{d\delta u}{dx} = \frac{1}{L} \delta u_{X2} \quad \Rightarrow$$

$$\delta w_{\Omega} = -\frac{1}{L} \delta u_{X2} EA \frac{1}{L} u_{X2} + \frac{x}{L} \delta u_{X2} \rho g A \quad \Rightarrow$$

$$\delta W = \int_0^L \delta w_{\Omega} dx = -\delta u_{X2} \left(EA \frac{1}{L} u_{X2} - \frac{1}{2} L \rho g A \right) \quad \Rightarrow$$

$$-\delta u_{X2} \left(EA \frac{1}{L} u_{X2} - \frac{1}{2} L \rho g A \right) = 0 \quad \Rightarrow \quad u_{X2} = \frac{1}{2} \frac{\rho g L^2}{E}.$$

The XZ-plane structure shown consists of two *massless* beams and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of displacements u_{Z2} and θ_{Y2}. Young's modulus of the beam material and the second moment of area are E and I, and the mass and moment of inertia of the disk are m and J, respectively.



Solution

4p The non-zero displacement/rotation components of the structure are u_{Z2} and θ_{Y2}. Let us start with the element contributions. Since the beam is assumed to be massless, only the virtual work expressions of the internal forces (available in the formulae collection) is needed.

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Element contribution of the rigid body (formulae collection) simplifies to

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} 0 \\ 0 \\ \ddot{u}_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ J\ddot{\theta}_{Y2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}.$$

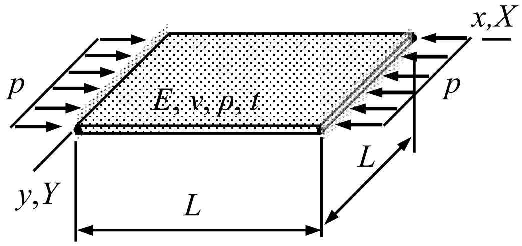
2p Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}\right).$$

Finally, principle of virtual work and the fundamental lemma of variation calculus imply a set of ordinary differential equations:

$$\frac{EI}{L^3} \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0.$$

Determine the critical value of the in-plane loading p_{cr} making the plate of the figure to buckle. The loaded edges are simply supported and the unloaded free. Use the approximation $w(x,y)=a_0(1-x/L)(x/L)$ and assume that $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$. Problem parameters E, ν, ρ and t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple in the linear analysis and that the in-plane stress resultants are known (from linear displacement analysis, say), it is enough to consider the virtual work densities of plate bending mode and the coupling of the bending and thin-slab modes

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix},\quad \delta w_\Omega^{\text{sta}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}$$

where the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

1p Approximation to the transverse displacement and its non-zero derivatives in the density expressions are given by

$$w(x,y)=a_0(1-\frac{x}{L})\frac{x}{L}\quad\Rightarrow\quad\frac{\partial w}{\partial x}=\frac{a_0}{L}(1-2\frac{x}{L}),\quad\frac{\partial^2 w}{\partial x^2}=-2\frac{a_0}{L^2}.$$

3p When the approximation is substituted there, virtual work density of the internal forces and that of the coupling simplify to (substitute also the known solution $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$ to the in-plane stress resultants)

$$\delta w_\Omega^{\text{int}}=-\delta a_0\frac{1}{3}\frac{t^3}{L^4}\frac{E}{1-\nu^2}a_0,$$

$$\delta w_\Omega^{\text{sta}}=\delta a_0(1-2\frac{x}{L})^2\frac{p}{L^2}a_0.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate

$$\delta W^{\text{int}}=\int_0^L\int_0^L\delta w_\Omega^{\text{int}}dxdy=-\delta a_0\frac{1}{3}\frac{t^3}{L^2}\frac{E}{1-\nu^2}a_0,$$

$$\delta W^{\text{sta}} = \int_0^L \int_0^L \delta w_\Omega^{\text{sta}} dxdy = \delta a_0 \frac{1}{3} p a_0 .$$

2p Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 (\frac{1}{3} \frac{t^3}{L^2} \frac{E}{1-\nu^2} - \frac{1}{3} p) a_0 ,$$

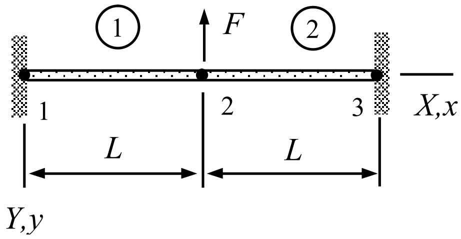
principle of virtual work $\delta W = 0 \; \forall \delta a_0$, and the fundamental lemma of variation calculus give

$$(\frac{1}{3} \frac{t^3}{L^2} \frac{E}{1-\nu^2} - \frac{1}{3} p) a_0 = 0 .$$

For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\text{cr}} = \frac{E}{1-\nu^2} \frac{t^3}{L^2} . \quad \leftarrow$$

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2} = 0$). When $F = 0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = CE_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx} = CE_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

2p For element 1, the non-zero displacement components is $u_{y2} = u_{Y2}$. As the initial length of the element $h^\circ = L$, linear approximations to the displacement components

$$u = w = 0 \text{ and } v = \frac{x}{L}u_{Y2} \quad \Rightarrow \quad \frac{du}{dx} = \frac{dw}{dx} = 0 \text{ and } \frac{dv}{dx} = \frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad \Rightarrow \quad \delta W^1 = -\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

2p For element 2, the non-zero displacement component $u_{y2} = u_{Y2}$. As the initial length of the element $h^\circ = L$, linear approximations to the displacement components

$$u = w = 0 \text{ and } v = (1 - \frac{x}{L})u_{Y2} \quad \Rightarrow \quad \frac{du}{dx} = \frac{dw}{dx} = 0 \text{ and } \frac{dv}{dx} = -\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad \Rightarrow \quad \delta W^2 = -\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

2p Virtual work expression of the point force is

$$\delta W^3 = -F\delta u_{Y2}.$$

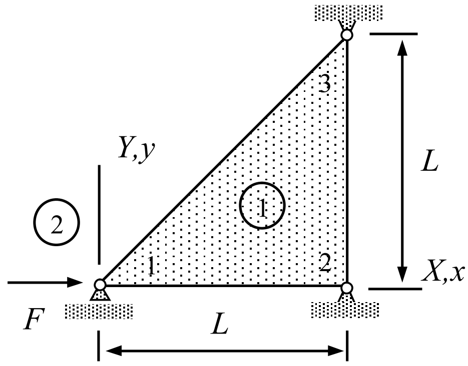
Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}[\frac{CA}{2}(\frac{u_{Y2}}{L})^3 + \frac{CA}{2}(\frac{u_{Y2}}{L})^3 + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(\frac{u_{Y2}}{L})^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -(\frac{FL^3}{CA})^{1/3}.$$

A thin triangular slab (assume plane stress conditions) loaded by a horizontal force is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. At the constant initial temperature ϑ° and loading $F=0$, stress vanishes. If the slab is heated to the constant temperature $2\vartheta^\circ$, what is the required force F to have $u_{X1}=0$? Material properties E, ν, α and thickness t of the slab are constants.



Solution

As temperature is known and the external distributed force vanishes, virtual work densities needed are (formulae collection)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ where } \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix},$$

$$\delta w_\Omega^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha t}{1-\nu} \Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

in which $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the difference between temperature at the deformed and initial geometries.

2p Approximation is the first thing to be considered. As the origin of the material $xy-$coordinate system is placed at node 1 and the axes are aligned with the axes of the structural $XY-$coordinate system

$$u = (1-\frac{x}{L})u_{X1}, \quad v=0, \text{ and } \Delta\vartheta = \vartheta^\circ \text{ (constant).}$$

2p When the approximations are substituted there, virtual work density (composed of the internal and coupling parts) simplifies to

$$\delta w_\Omega = -\begin{Bmatrix} -\delta u_{X1}/L \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -u_{X1}/L \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -\delta u_{X1}/L \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu} \alpha\vartheta^\circ \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_\Omega = -\frac{\delta u_{X1}}{L} \frac{Et}{1-\nu^2} \frac{u_{X1}}{L} - \frac{\delta u_{X1}}{L} \frac{Et}{1-\nu} \alpha\vartheta^\circ .$$

Virtual work expression is integral of the density expression over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area. Virtual work expressions of element 1 and 2 (point force) become

$$\delta W^1 = \delta w_\Omega \frac{L^2}{2} = -\delta u_{X1}(\frac{1}{2}\frac{Et}{1-\nu^2} u_{X1} + \frac{1}{2}\frac{Et}{1-\nu} L\alpha\vartheta^\circ),$$

$\delta W^2 = \delta u_{X1} F$.

2p Virtual work expression of the structure $\delta W = \delta W^1 + \delta W^2$, principle of virtual work, and the fundamental lemma of variation calculus imply the equilibrium equation

$$\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1} + \frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^\circ - F = 0 \quad \Leftrightarrow \quad u_{X1} = \frac{1-\nu^2}{Et}(2F - \frac{Et}{1-\nu}L\alpha\vartheta^\circ) . \quad \leftarrow$$

Displacement vanishes with the force (this is also the horizontal constraint force when the node is fixed)

$$F = \frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^\circ . \quad \leftarrow$$